An extension the Liénard–Wiechert retardation equations to include the Thomas precession

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If a Lorentz transform is performed to the frame of reference of an accelerated particle at time t, followed by an infinitesimal transform to the velocity of the particle at time $t + dt$, the result is the same as a direct transform at time $t+dt$, followed by a space rotation. This relationship is the basis of the Thomas precession. The 4-potential transforms in the same way as the coordinates, so the rotation also affects the vector potential. The calculations of the Thomas precession are extended to the third time derivative, then used to obtain the retardation equations for a charged particle. The solutions have to be integrated in time in order to obtain the potential solution. The integration is always easy for periodic solutions, however the static terms are lost.

I. INTRODUCTION

The tensor of each rank is irreducible[2], but there are no consequences of the irreducibility if the differentiations are performed in the first frame of reference, as the integral of the differential restores the original.

When the differentiations are performed in the frame of reference of the particle, then the derivatives of the retarded potentials integrated in the frame of reference of the field point, the irreducibility theorem becomes applicable. The result is that the solutions for a tensor of a given rank are not representable with linear combinations of the solutions of lower rank tensors.

II. A COMPUTER ALGORITHM

The following calculations are a continuation of the relationships investigated in Ref. 4, and similar calculations are shown in more detail there.

The Taylor expansion of the trajectory is

$$\begin{aligned} \boldsymbol{r}_{\rm s} &= \boldsymbol{r_0} + \boldsymbol{v}\delta t_{\rm s} + \boldsymbol{a}(\delta t_{\rm s})^2/2 + \dot{\boldsymbol{a}}(\delta t_{\rm s})^3/6 + \ddot{\boldsymbol{a}}(\delta t_{\rm s})^4/24 \\ t_{\rm s} &= -r_0/c + \delta t_{\rm s}. \end{aligned}$$

$t_{\rm s}$ is the time at the particle, as evaluated in the first frame of reference. The substitution $\delta t_{\rm s} = \delta t_{\rm s} + dt_{\rm s}$ must be made in the equation, with $dt_{\rm s}$ only being carried to the first power. $dt_{\rm s}$ is used for computing the velocity for the transforms. It is set to zero when imposing the light cone condition. The coordinates of the field point are

$$\begin{aligned} \boldsymbol{r}_{\rm f} &= 0 \\ t_{\rm f} &= \delta t_{\rm f} + dt_{\rm f}. \end{aligned}$$

$dt_{\rm f}$ is only used for computing the velocity of the final transform. It is set to zero when imposing the light cone condition.

The method of successive approximation is used to solve the light cone equation for $\delta t_{\rm s}$ in terms of $\delta t_{\rm f}$, then all occurrences of $\delta t_{\rm s}$ are replaced by $\delta t_{\rm f}$. The maximum power of $\delta t_{\rm s}$ carried when solving the light cone equation must not be independently constrained. Very large powers of $\delta t_{\rm s}$ occur when the velocity is high.

The solution for $\delta t_{\rm s}$ is used to construct an array of events in the first frame of reference. Neglecting the $-r/c$ terms, the times of the particle events, as parameterized by $\delta t_{\rm f}$, are dts, $\delta t_{\rm f}/n + dts$, ... $\delta t_{\rm f} + dts$. The times of the events at the field point are $dt_{\rm f}$, $\delta t_{\rm f}/n + dt_{\rm f}$, ... $\delta t_{\rm f} + dt_{\rm f}$. The first difference of the potentials during the interval $k\delta t_{\rm f}/n$ to $(k+1)\delta t_{\rm f}/n$ will be needed, with $0 \leq k < n$.

The velocity v_{12} of the particle at time $k\delta t_{\rm f}/n$ is easily obtained, then it is used to transform the locations of both the field point and the particle to the second frame of reference. The locations at time $(k+1)\delta t_{\rm f}/n$ must also be transformed with the same velocity.

To first order, the particle is at rest in the second frame of reference at time $k\delta t_{\rm f}/n$. The potentials at the field point are $\boldsymbol{A} = 0$, $\psi = q/r$. They are transformed back to the first frame of reference with the velocity $-v_{12}$ to obtain the Liénard–Wiechert (LW) solution for that time.

In the second frame of reference, the change in the particle's position during the $dt_{\rm s}$ interval at time $(k+1)\delta t_{\rm f}/n$ is obtained by setting $dt_{\rm s}$ to 0 then subtracting it from the space part of the full solution. The change in the time at the particle during the $dt_{\rm s}$ interval, as parameterized by $\delta t_{\rm f}$, is obtained in the same way for the time part of the equation. The ratio of space and time parts is v_{23}. The lowest order term of v_{23} is $\boldsymbol{a}\delta t_{\rm f}/n$. This velocity is used to transform the locations of the particle and the field point at time $(k+1)\delta t_{\rm f}/n$ to the third frame of reference. $dt_{\rm s}$ drops out of the space part of the solution, showing that the particle has been halted in the third frame of reference.

The potentials are computed as before, then the velocity of the field point in the third frame of reference at time $(k+1)\delta t_{\rm f}/n)$ is computed in the same way as the v_{23} velocity, except that $dt_{\rm f}$ is used instead of $dt_{\rm s}$. The potentials are then transformed directly back to the first frame of reference. This path includes the Thomas precession[1,3]. In potential form, the solution contains is the sum of the LW and Thomas terms at time $(k+1)\delta t_{\rm f}/n$. Subtracting the LW solution for the time $k\delta t_{\rm f}/n$ provides the first difference of the sum.

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The LW solution for the time $(k+1)\delta t_\mathrm{f}/n$ is then independently computed in the same way as it was for earlier time. Subtracting the LW solution for the earlier time leaves the first difference of the LW terms. Then subtracting the result from the first difference of the sum of the terms isolates the Thomas terms. The equations are linear, so it is not necessary to carry the LW terms in order to analyze the behavior of the Thomas terms.

On the next pass through the loop, with k now being one more than before, the solution for the LW terms at time $(k+1)\delta t_\mathrm{f}/n$ of the prior pass is now the solution for the time $k\delta t_\mathrm{f}/n$, so it can be re-used to save processing time.

This calculation, along with source code listings and some related derivations, is shown in in the supplemental online material (SOM) at www.s-4.com/som2. Most of the online material is in the form of raw computer output.

III. THE THOMAS TERMS

The first differences of the arrays can be differenced to obtain the second differences, then differenced again to obtain the third differences. The equation for converting the third difference to the third derivative will be needed. The tensor of the fourth rank represents the third derivatives.

In order to represent the equation $A = t^3$ with three straight-line segments, the solution for four equally spaced times is needed, with those being [0, $1^3/3^3$, $2^3/3^3$, 1]. The first difference is [1/27, 7/27, 19/27]. The second difference is [2/9, 4/9], and the third difference is 2/9.

All of the third differences are the same when more than three line segments are used. It is shown in the SOM that the third difference for n segments is $6/n^3$.

The third term of a Taylor extrapolation is $(d^3A/dt^3)t^3/6$. In the above equation d^3A/dt^3 is 6. It is required that extrapolating the third difference in the absence of the Thomas terms works in the same way as the Taylor theorem, so the analogue of the Taylor equation becomes $n^3\delta_3 t^3/6$, where δ_3 represents the third difference.

All of the third differences provided by the above algorithm are the same, so the procedure for obtaining the retardation equations is to select one of them and multiply it by n^3. The solution does not depend on the value of n, provided that it is at least 3. $(\delta t_\mathrm{f})^3$ is set to 1 after all of the calculations are complete. There are no other powers of δt_f in the third difference. It extrapolates in the same way as the Taylor equation, and the extrapolation remains valid for arbitrarily large values of n, so the solution represents $d^3\boldsymbol{A}_\mathrm{T}/dt^3$. The space rotation of the Thomas precession does not affect the scalar potential, so it is the same as for the LW solution.

The LW terms can be included in the calculation, but the third derivative of the LW solution with respect to δt_f provides the same solution, and it is easier to obtain. The full solution is the sum of the LW and Thomas terms. Except under extreme conditions, the Thomas terms are weak in relation to the LW terms.

The solution to order v^7, a^3, $\dot{a}^2$, and $\ddot{a}^1$ is shown in the SOM. There are higher powers of velocity in all orders, but the higher powers of the other terms drop out.

The solution for the third rank tensor is obtained by dropping the $\ddot{\boldsymbol{a}}$ terms in the trajectory and taking the second difference. The conversion factor for the second difference is n^2. The solution to order v^{12}, a^2, and $\dot{a}^1$ is shown in the SOM. There are no a^3, $\dot{a}^2$, or $\ddot{a}$ terms in the second derivative. There are higher powrers of velocity in all orders, but the higher powers are not meaningful in many solutions if the $\ddot{\boldsymbol{a}}$ terms are not carried.

The solution for the second rank tensor is obtained by dropping both the $\dot{\boldsymbol{a}}$ and $\ddot{\boldsymbol{a}}$ terms of the trajectory. The exact solution for the first derivative was obtained in Ref. 4.

The time derivatives of the retarded potentials have a worthwhile utility, but the static terms are lost when integrating the time derivatives of periodic solutions, so the space derivatives will be needed for retaining them.

Some plausibility tests of the retardation equations are developed in the SOM. The solutions appear to be well behaved, and reduce to the LW result in the expected ways. For a single charged particle in a circular orbit, the retardation solution for the fourth rank tensor satisfies two of the Maxwell equations. It satisfies three of the four restructured Proca equations that were obtained in Ref 4. The complete set of field equations for the fourth rank tensor has not yet been identified.

The solutions for the third and fourth rank tensors contain near field magnetic terms that may be suitable for quasi-static laboratory investigations if the Thomas terms prove to be separable from the much larger Maxwellian terms. Separability is more important than the magnitude of the fields, as refined electrical detection methods are extremely sensitive. Rotating equipment will be required, as the Thomas terms are far too weak to detect in stationary configurations. The space derivatives are better suited to obtaining quasi-static solutions.

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[3] J. D. Jackson, *Classical Electrodynamics*, (John Wiley and Sons, New York, 1975)
[4] G. Osborn, *A low order extension the Liénard–Wiechert retardation equations to include the Thomas precession* (arXiv:1409.2101 [gr-qc])