A generalization of the Thomas precession, Part I

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The time shown by an accelerated clock depends on its history. The Lorentz transform does not distinguish between the history of an accelerated clock and a constant velocity clock. The history of the clock can be assimilated by integrating the first derivative of the Thomas precession from the time $t = 0$. A definite integral is required because the unknown trajectory of the clock in the distant past affects its displayed time. The coordinates are spinning in the second frame of reference during the integration, but in the definite integral from time $t = 0$ to time t the spin accumulates to a specific angle. The integral is equivalent to a Lorentz transform followed by a space rotation. A space rotation does not affect the invariant quantity $r^2 - c^2t^2$. The history of a jerked clock is different than that of an accelerated clock. The solution in that order is equivalent to a Lorentz transform followed by two consecutive space rotations in different directions. Similarly, there are three rotations in the $\ddot{\boldsymbol{a}}$ solution.

I. OVERVIEW

The following calculations are mathematically interesting, but I am not sure what the mean. Invariance is a necessary but not sufficient condition. Invariant solutions are not necessarily of physical significance. The calculations do illustrate that the Lorentz transform is not unique in holding the speed of light invariant. Other invariant solutions may exist.

II. THE TAYLOR THEOREM

The trajectory of a particle can be expanded as a Taylor series

$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{a}_0 t^2 + \frac{1}{6}\dot{\boldsymbol{a}}_0 t^3 + \frac{1}{24}\ddot{\boldsymbol{a}}_0 t^4 + \ \ldots \ . \quad (1)$$

Eq. (1) does not represent an equation of motion. A clock moves along a marked course, and the time in the equation is the time shown by the nearest at-rest clock. The moving clock is synchronized to the time shown by the nearest stationary clock at the time $t = 0$. Thereafter it runs slower than the stationary clock, but the time in the first frame of reference remains the time shown by the nearest clock in a field of clocks. The acceleration terms in Eq. (1) are not real in a physical sense.

In the multivariate Taylor theorem[2] the sum of the exponents in each term is the same. Terms not in accord with the theorem occur routinely in series calculations. The terms are usually harmless, but they should be dropped in the interest of computational efficiency. It is all right to carry selected variables to a lower power than is allowed by the Taylor theorem. dt is only used for computing the derivatives, so it does not count as one of the multivariate variables.

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III. THE ACCELERATED CLOCK

The Lorentz transform in vector form is

$$\begin{aligned} \boldsymbol{r}' &= \gamma(\boldsymbol{r} - \boldsymbol{v}t) - (\gamma - 1)(\boldsymbol{r} - \boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{r}/v^2) \\ t' &= \gamma(t - \boldsymbol{v}\cdot\boldsymbol{r}/c^2), \end{aligned}$$

with $\gamma = (1 - v^2/c^2)^{-\frac{1}{2}}$. The infinitesimal transform is obtained by setting γ to 1

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{r} - \boldsymbol{v}t \\ t' &= t - \boldsymbol{v}\cdot\boldsymbol{r}/c^2. \end{aligned} \quad (2)$$

To order v^3, the full transform in series form is

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{r} - \boldsymbol{v}t(1 + \frac{v^2}{2c^2}) + \frac{1}{2c^2}\boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{r} \\ t' &= t(1 + \frac{v^2}{2c^2}) - \frac{1}{c^2}\boldsymbol{v}\cdot\boldsymbol{r} - \frac{v^2}{2c^4}\boldsymbol{v}\cdot\boldsymbol{r} \end{aligned} \quad (3)$$

The calculations in this section are of too low an order to be of much interest, but they illustrate the behavior of the equations, and more complete calculations are shown in the supplemental online material (SOM) at `www.s-4.com/xfm`. A more recent version of this paper may also be available there. This PDF document is also available at `http://vixra.org/abs/1708.0090`, but there is no supplemental data at that site.

The $\dot{\boldsymbol{a}}$ and $\ddot{\boldsymbol{a}}$ terms in Eq. (1) can be set to zero for observations of short duration

$$\begin{aligned} r_i &= \boldsymbol{r}_0 + \boldsymbol{v}_0(t + \ dt) + \boldsymbol{a}_0(t + dt)^2/2 \\ &= \boldsymbol{r}_0 + \boldsymbol{v}_0(t + \ dt) + \boldsymbol{a}_0 t^2/2 + \boldsymbol{a}_0 t \ dt \\ t_i &= t + dt. \end{aligned}$$

The remaining terms can be represented as three straight-line segments in a coarse approximation of the trajectory. The four points on the trajectory needed to define three segments are obtained by substituting $t = 0$,

$t = t/3$, $t = 2/3\ t$, and $t = t$ into these equations

$$\begin{aligned}
\boldsymbol{r}_0 &= \boldsymbol{r}_0 + \boldsymbol{v}_0\ dt \\
t_0 &= dt \\
\boldsymbol{r}_1 &= \boldsymbol{r}_0 + \boldsymbol{v}_0 t/3 + \boldsymbol{v}_0\ dt + \boldsymbol{a}_0 t\ dt/3 + \boldsymbol{a}_0 t^2/18 \\
t_1 &= t/3 +\ dt \\
\boldsymbol{r}_2 &= \boldsymbol{r}_0 + 2/3\ \boldsymbol{v}_0 t + \boldsymbol{v}_0\ dt + 2/3\ \boldsymbol{a}_0 t\ dt + 2/9\ \boldsymbol{a}_0 t^2 \\
t_2 &= +2/3\ t + dt \\
\boldsymbol{r}_3 &= \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \boldsymbol{v}_0\ dt + \boldsymbol{a}_0 t\ dt + \boldsymbol{a}_0 t^2/2 \\
t_3 &= t + dt.
\end{aligned}$$

All of the points are now transformed to the second frame of reference with the velocity $\boldsymbol{v}_{12} = \boldsymbol{v}_0$ and Eq. (3). These calculations are to order v^2a^1. The v^2a terms are in the same order as the v^3 terms, so they are dropped. (The v^3 terms are shown in Eq. (3), but they are not used here).

$$\begin{aligned}
\boldsymbol{r}'_0 &= \boldsymbol{r}_0 + \boldsymbol{v}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/(2c^2) \\
t'_0 &= -\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/c^2 + dt - v_0^2\ dt/(2c^2) \\
\boldsymbol{r}'_1 &= \boldsymbol{r}_0 + \boldsymbol{v}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/(2c^2) + \boldsymbol{a}_0 t\ dt/3 + \boldsymbol{a}_0 t^2/18 \\
t'_1 &= t/3 + dt - \boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/c^2 - v_0^2\ dt/(2c^2) - tv_0^2/(6c^2) \\
&\quad -t\boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0\ dt/(3c^2) - t^2\boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0/(18c^2) \\
\boldsymbol{r}'_2 &= \boldsymbol{r}_0 + \boldsymbol{v}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/(2c^2) + 2/3\ \boldsymbol{a}_0 t\ dt + 2/9\ \boldsymbol{a}_0 t^2 \\
t'_2 &= 2/3\ t + dt - v_0^2\ dt/(2c^2) - tv_0^2/(3c^2) - \boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/c^2 \\
&\quad -2/3\ \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t\ dt/c^2 - 2/9\ \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t^2/c^2 \\
\boldsymbol{r}'_3 &= \boldsymbol{r}_0 + \boldsymbol{v}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/(2c^2) + \boldsymbol{a}_0 t\ dt + \boldsymbol{a}_0 t^2/2 \\
t'_3 &= t + dt - v_0^2\ dt/(2c^2) - v_0^2 t/(2c^2) - \boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/c^2 \\
&\quad -\boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t\ dt/c^2 - \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t^2/(2c^2)
\end{aligned}$$

dt has dropped out of the location of the clock at time t_0, showing that it has been halted in the second frame of reference. The clock is accelerated, so it acquires a velocity at time $t_1 = t/3$. The clock must now be halted in the third frame of reference.

The velocity for the second transform is $d\boldsymbol{r}'/dt' = [\boldsymbol{r}'_1(dt) - \boldsymbol{r}'_1(0)]/[t'_1(dt) - t'_1(0)] = \frac{1}{3}\boldsymbol{a}_0 t/[1 - \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t/(3c^2) - v_0^2/(2c^2)]$. There is an a_0^2 term in the series expansion of this solution, but it cannot be consistently carried without also carrying the $\dot{\boldsymbol{a}}$ terms, so it has to be dropped. (The a^2 terms are small when the motion is gentle.) The v^2a terms are also not being carried in the calculations of this order, so the velocity for the transform to the third frame of reference simplifies to $\boldsymbol{a}_0 t/3$. The infinitesimal transform in Eq. (2) is now used to transform $\boldsymbol{r}'_1$, t'_1, and all following points to the third frame of reference.

$$\begin{aligned}
\boldsymbol{r}'_1 &= \boldsymbol{r}_0 + \boldsymbol{v}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/(2c^2) + \boldsymbol{a}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0 t/(3c^2) - \boldsymbol{a}_0 t^2/18 \\
t'_1 &= t/3 + dt - v_0^2\ dt/(2c^2) - v_0^2 t/(6c^2) - \boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/c^2 \\
&\quad -\boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{r}_0 t/(3c^2) - \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t\ dt/(3c^2) \\
&\quad -\boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t^2/(18c^2) \\
\boldsymbol{r}'_2 &= \boldsymbol{r}_0 + \boldsymbol{v}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/(2c^2) + \boldsymbol{a}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0 t/(3c^2) \\
&\quad +\boldsymbol{a}_0 t\ dt/3 \\
t'_2 &= 2/3\ t + dt - v_0^2\ dt/(2c^2) - v_0^2 t/(3c^2) - \boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/c^2 \\
&\quad -\boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{r}_0 t/(3c^2) - 2/3\ \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t\ dt/c^2 \\
&\quad -2/9\ \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t^2/c^2 \\
\boldsymbol{r}'_3 &= \boldsymbol{r}_0 + \boldsymbol{v}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/(2c^2) + \boldsymbol{a}_0\boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0 t/(3c^2) \\
&\quad +2/3\ \boldsymbol{a}_0 t\ dt + \boldsymbol{a}_0 t^2/6 \\
t'_3 &= t + dt - v_0^2\ dt/(2c^2) - v_0^2 t/(2c^2) - \boldsymbol{v}_0\boldsymbol{\cdot}\boldsymbol{r}_0/c^2 \\
&\quad -\boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{r}_0 t/(3c^2) - \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t\ dt/c^2 - \boldsymbol{a}_0\boldsymbol{\cdot}\boldsymbol{v}_0 t^2/(2c^2)
\end{aligned}$$

dt has dropped out of the location of the clock at time t_1, showing that, to first order, it is at rest in the third frame of reference.

The coordinates are now reparameterized by the coordinates first-known at time t_1 by the equations

$$\begin{aligned}
\boldsymbol{r}_0 &= \boldsymbol{r}_1 - \boldsymbol{v}_1 t/n \\
\boldsymbol{v}_0 &= \boldsymbol{v}_1 - \boldsymbol{a}_1 t/n \\
\boldsymbol{a}_0 &= \boldsymbol{a}_1 - \dot{\boldsymbol{a}}_1 t/n.
\end{aligned} \tag{4}$$

n is 3 in this example, but it would have to be much larger for the equations to be accurate. The equations become exact as n goes to ∞, except that the $\ddot{\boldsymbol{a}}$ terms have been set aside for now. $\dot{\boldsymbol{a}}$ is also assumed to be small enough that it can be neglected, so $\boldsymbol{a}_0 = \boldsymbol{a}_1$. Then

$$\begin{aligned}
\boldsymbol{r}'_2 &= \boldsymbol{r}_1 - \boldsymbol{v}_1 t/3 + \boldsymbol{v}_1\boldsymbol{v}_1\boldsymbol{\cdot}\boldsymbol{r}_1/(2c^2) - \boldsymbol{v}_1\boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{r}_1 t/(6c^2) \\
&\quad +\boldsymbol{a}_1\boldsymbol{v}_1\boldsymbol{\cdot}\boldsymbol{r}_1 t/(6c^2) + \boldsymbol{a}_1 t\ dt/3 \\
t'_2 &= 2/3\ t + dt - v_1^2\ dt/(2c^2) - \boldsymbol{v}_1\boldsymbol{\cdot}\boldsymbol{r}_1/c^2 \\
&\quad -\boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{v}_1 t\ dt/(3c^2) \\
\boldsymbol{r}'_3 &= \boldsymbol{r}_1 - \boldsymbol{v}_1 t/3 + \boldsymbol{v}_1\boldsymbol{v}_1\boldsymbol{\cdot}\boldsymbol{r}_1/(2c^2) - \boldsymbol{v}_1\boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{r}_1 t/(6c^2) \\
&\quad +\boldsymbol{a}_1\boldsymbol{v}_1\boldsymbol{\cdot}\boldsymbol{r}_1 t/(6c^2) + 2/3\ \boldsymbol{a}_1 t\ dt + \boldsymbol{a}_1 t^2/6 \\
t'_3 &= t + dt - v_1^2\ dt/(2c^2) - v_1^2 t/(6c^2) - \boldsymbol{v}_1\boldsymbol{\cdot}\boldsymbol{r}_1/c^2 \\
&\quad -2/3\ \boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{v}_1 t\ dt/c^2 - \boldsymbol{a}_1\boldsymbol{\cdot}\boldsymbol{v}_1 t^2/(6c^2).
\end{aligned}$$

Because the $\dot{\boldsymbol{a}}$ terms are being neglected, the velocity for the third transform simplifies to the same form it had for the second transform, $\boldsymbol{a}_1 t/3$. Transforming all of the remaining points to the fourth frame of reference with the infinitesimal transform,

$$\begin{aligned}
\boldsymbol{r}_2' &= \boldsymbol{r}_1 - \boldsymbol{v}_1 t/3 + \boldsymbol{v}_1\boldsymbol{v}_1\cdot\boldsymbol{r}_1/(2c^2) + \boldsymbol{a}_1\boldsymbol{v}_1\cdot\boldsymbol{r}_1 t/(2c^2)\\
&\quad -\boldsymbol{v}_1\boldsymbol{a}_1\cdot\boldsymbol{r}_1 t/(6c^2) - 2/9\ \boldsymbol{a}_1 t^2\\
t_2' &= 2/3\ t + dt - v_1^2\ dt/(2c^2) - \boldsymbol{v}_1\cdot\boldsymbol{r}_1/c^2\\
&\quad -\boldsymbol{a}_1\cdot\boldsymbol{r}_1 t/(3c^2) - \boldsymbol{a}_1\cdot\boldsymbol{v}_1 t\ dt/(3c^2)\\
&\quad +\boldsymbol{a}_1\cdot\boldsymbol{v}_1 t^2/(9c^2)\\
\boldsymbol{r}_3' &= \boldsymbol{r}_1 - \boldsymbol{v}_1 t/3 + \boldsymbol{v}_1\boldsymbol{v}_1\cdot\boldsymbol{r}_1/(2c^2) + \boldsymbol{a}_1 t\ dt/3\\
&\quad +\boldsymbol{a}_1\boldsymbol{v}_1\cdot\boldsymbol{r}_1 t/(2c^2) - \boldsymbol{v}_1\boldsymbol{a}_1\cdot\boldsymbol{r}_1 t/(6c^2) - \boldsymbol{a}_1 t^2/6\\
t_3' &= t + dt - v_1^2\ dt/(2c^2) - v_1^2 t/(6c^2) - \boldsymbol{v}_1\cdot\boldsymbol{r}_1/c^2\\
&\quad -\boldsymbol{a}_1\cdot\boldsymbol{r}_1 t/(3c^2) - 2/3\ \boldsymbol{a}_1\cdot\boldsymbol{v}_1 t\ dt/c^2\\
&\quad -\boldsymbol{a}_1\cdot\boldsymbol{v}_1 t^2/(18c^2).
\end{aligned}$$

Eqs. (4) are now used to reparametrize the solution by coordinates first-known at time t_2

$$\begin{aligned}
\boldsymbol{r}_3' &= \boldsymbol{r}_2 - 2/3\ \boldsymbol{v}_2 t + \boldsymbol{v}_2\boldsymbol{v}_2\cdot\boldsymbol{r}_2/(2c^2) - \boldsymbol{v}_2\boldsymbol{a}_2\cdot\boldsymbol{r}_2 t/(3c^2)\\
&\quad +\boldsymbol{a}_2 t\ dt/3 + \boldsymbol{a}_2\boldsymbol{v}_2\cdot\boldsymbol{r}_2 t/(3c^2) - \boldsymbol{a}_2 t^2/18\\
t_3' &= t + dt - v_2^2\ dt/(2c^2) + v_2^2 t/(6c^2) - \boldsymbol{v}_2\cdot\boldsymbol{r}_2/c^2\\
&\quad -\boldsymbol{a}_2\cdot\boldsymbol{v}_2 t\ dt/(3c^2) + \boldsymbol{a}_2\cdot\boldsymbol{v}_2 t^2/(18c^2).
\end{aligned}$$

The velocity for the transform to the fifth and last frame of reference is $\boldsymbol{a}_2 t/3$

$$\begin{aligned}
\boldsymbol{r}_3' &= \boldsymbol{r}_2 - 2/3\ \boldsymbol{v}_2 t + \boldsymbol{v}_2\boldsymbol{v}_2\cdot\boldsymbol{r}_2/(2c^2)\\
&\quad +2/3\ \boldsymbol{a}_2\boldsymbol{v}_2\cdot\boldsymbol{r}_2 t/c^2 - \boldsymbol{v}_2\boldsymbol{a}_2\cdot\boldsymbol{r}_2 t/(3c^2) - 7/18\ \boldsymbol{a}_2 t^2\\
t_3' &= t + dt - v_2^2\ dt/(2c^2) + v_2^2 t/(6c^2) - \boldsymbol{v}_2\cdot\boldsymbol{r}_2/c^2\\
&\quad -\boldsymbol{a}_2\cdot\boldsymbol{r}_2 t/(3c^2) - \boldsymbol{a}_2\cdot\boldsymbol{v}_2 t\ dt/(3c^2)\\
&\quad +5/18\ \boldsymbol{a}_2\cdot\boldsymbol{v}_2 t^2/c^2.
\end{aligned}$$

The solution is reparameterized for the last time as in Eqs. (4). t and t_3 are the same, so the subscripts can be dropped. dt will no longer be needed, so it can be set to zero. Then

$$\begin{aligned}
\boldsymbol{r}' &= \boldsymbol{r} - \boldsymbol{v}t - \boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{r}t/(2c^2) + \boldsymbol{a}\boldsymbol{v}\cdot\boldsymbol{r}t/(2c^2)\\
&\quad +\boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{r}/(2c^2) - \boldsymbol{a}t^2/6\\
t' &= t + v^2 t/(2c^2) - \boldsymbol{v}\cdot\boldsymbol{r}/c^2 + \boldsymbol{a}\cdot\boldsymbol{v}t^2/(6c^2).
\end{aligned}$$

This algorithm requires approximately $\frac{1}{2}n^2$ transforms. The $-\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{r}t$ and $+\boldsymbol{a}\boldsymbol{v}\cdot\boldsymbol{r}t$ terms represent the cumulative effect of the Thomas precession from the time $t = 0$ to the time t. These two terms cancel each other when $\boldsymbol{a}$ and $\boldsymbol{v}$ are parallel. The coordinates are spinning, but the spin is not visible in the first frame of reference until the solution is differentiated with respect to t.

Repeating the calculation with more steps and more powers of velocity shows that some coefficients do not depend on the value of n, while the residuals of others vary precisely as $1/n$. Some coefficients go to zero as n goes to ∞, while others approach a specific value. The asymptotic solution is obtainable by extrapolating each coefficient with the equation

$$C(\infty) = C(n+1) + n[C(n+1) - C(n)]. \qquad (5)$$

Within the limitations of a calculation in series form, the extrapolation is exact for any power of velocity and any $n \geq 1$. The acceleration solution is highly degenerate. It is obtainable with an infinity of infinitesimal Lorentz transforms, but the method is an overkill. The simplicity of the relationships implies that a closed form solution exists. It would probably be obtainable with the method of undetermined coefficients.

From the SOM, the solution to order v^4a^1 is

$$\begin{aligned}
\boldsymbol{r}' &= \boldsymbol{r} - \boldsymbol{v}t + \frac{3v^2}{8c^4}\boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{r} - \frac{v^2}{2c^2}\boldsymbol{v}t + \frac{1}{2c^2}\boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{r}\\
&\quad + \frac{5v^2}{8c^4}\boldsymbol{a}\boldsymbol{v}\cdot\boldsymbol{r}t + \frac{1}{2c^2}\boldsymbol{a}\boldsymbol{v}\cdot\boldsymbol{r}t - \frac{v^2}{2c^2}\boldsymbol{a}t^2\\
&\quad - \frac{1}{4c^4}\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{v}\boldsymbol{v}\cdot\boldsymbol{r}t - \frac{3v^2}{8c^4}\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{r}t - \frac{1}{2c^2}\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{r}t\\
&\quad + \frac{1}{2c^2}\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{v}t^2\\
t' &= t - \frac{1}{c^2}\boldsymbol{v}\cdot\boldsymbol{r} + \frac{v^2}{2c^2}t - \frac{v^2}{2c^4}\boldsymbol{v}\cdot\boldsymbol{r} + \frac{3v^4}{8c^4}t.
\end{aligned}$$

The invariant quantity, $\boldsymbol{r}\cdot\boldsymbol{r} - c^2t^2$, is $\boldsymbol{r}\cdot\boldsymbol{r} - c^2t^2$. The acceleration terms have no effect on it. The Lorentz transform contains a hidden degree of freedom. Solutions that are superficially unique are actually a family of solutions. The member of the family that is selected by the equations is not necessarily the right one for a global solution. The solution for a smaller family of particles can be obtained by carrying more terms in the equations. Even the Newton equations are not free of this ambiguity. The solution at time $dt_1 + dt_2$ is actually for a family of particles, but that does not matter for the Newton equations. It does matter in a four dimensional space.

Since the time part is the same as the Lorentz transform and the magnitude of the vector is also the same as the Lorentz transform, the transform is equivalent to a Lorentz transform followed by a space rotation. The transform reduces to the Lorentz transform when the $\boldsymbol{a}$ and $\boldsymbol{v}$ vectors are parallel or anti-parallel, which is the signature of the Thomas precession[1,4]. The solution to order $v^{25}a^1$ is shown in the SOM.

IV. THE JERKED CLOCK

The $\dot{\boldsymbol{a}}$ terms integrate to acceleration terms, resulting in a second rotation later in the trajectory. In general, $\boldsymbol{a}$ and $\dot{\boldsymbol{a}}$ are in independent directions.

In the second frame of reference the a^2 terms are in the same order as the $\dot{\boldsymbol{a}}$ terms, so they have to be carried, but the a^3 and $\dot{a}^2$ terms drop out unless the $\ddot{\boldsymbol{a}}$ terms are also carried.

If a curve is approximated as a series of n straight line segments, the residual in each segment varies as $1/n^2$. When integrating numerically in n steps, the sum of the errors varies approximately as $1/n$. The Eq. (5) extrapolation will take out the $1/n$ terms, but there are also $1/n^2$ and other terms in most equations.

Since the software execution time varies approximately as $\frac{1}{2}n^2$ for large n, it much more efficient to perform two calculations for n and $n+1$ steps, then extrapolate the solutions, than to perform one calculation with the same accuracy and a larger n. It is still more efficient to extrapolate the solutions for n, $n+1$, and $n+2$ steps, and a third order extrapolation is even better.

The extrapolation formulas accomplish more than simply performing the integration with more line segments, because the residuals generally approach $1/n$ as n becomes large, then the last extrapolation will sometimes take out the final $1/n$ terms altogether.

The second order extrapolation equation is

$$\begin{aligned} C_{12} &= C(n+1) + n[C(n+1) - C(n)] \\ C_{23} &= C(n+2) + (n+1)[C(n+2) - C(n+1)] \\ C(\infty) &= C_{23} + \frac{n}{2}[C_{23} - C_{12}] \end{aligned}$$

This extrapolation is only exact if there are no $1/n^3$ terms in the equation, and most equations do have $1/n^3$ terms. The extrapolation is a good approximation when it is not exact, and it becomes better when n is large, because the $1/n^3$ terms are then less important.

The extrapolation formulas are elementary, and they are derived in the SOM. They are probably not of continuing interest for this problem, but they provide an expedient way of obtaining solutions. They represent a general purpose algorithm and they do not reveal much about the geometry of the problem.

To order v^4, the fifth order extrapolation equation yields an exact result for a jerked clock, for any $n \geq 1$. Since the solution does not depend on the value of n, the calculations exist in the first order of the infinitesimal. Infinitesimal quantities can be subdivided, but it does not accomplish anything. The relationships are linearly dependent and the final solution remains the same.

To order $v^4 a^2 \dot{a}^1$ the solution is

$$\begin{aligned}
&\boldsymbol{r}' = \boldsymbol{r} + 5/8\,\boldsymbol{a}tv^2\boldsymbol{r}\cdot\boldsymbol{v}/c^4 - \boldsymbol{a}t^2v^2\boldsymbol{a}\cdot\boldsymbol{r}/(8c^4) - \\
&\boldsymbol{a}t^2\boldsymbol{a}\cdot\boldsymbol{v}\boldsymbol{r}\cdot\boldsymbol{v}/(4c^4) - \boldsymbol{a}t^3v^2\dot{\boldsymbol{a}}\cdot\boldsymbol{r}/(48c^4) + \boldsymbol{a}t^3\dot{\boldsymbol{a}}\cdot\boldsymbol{v}\boldsymbol{r}\cdot\boldsymbol{v}/(24c^4) - \\
&\boldsymbol{a}t^4\boldsymbol{a}\cdot\dot{\boldsymbol{a}}\boldsymbol{r}\cdot\boldsymbol{v}/(16c^4) + \boldsymbol{a}t^4\boldsymbol{a}\cdot\boldsymbol{r}\dot{\boldsymbol{a}}\cdot\boldsymbol{v}/(24c^4) + \\
&\boldsymbol{a}t^4\boldsymbol{a}\cdot\boldsymbol{v}\dot{\boldsymbol{a}}\cdot\boldsymbol{r}/(12c^4) + \boldsymbol{a}t\boldsymbol{r}\cdot\boldsymbol{v}/(2c^2) - \boldsymbol{a}t^2v^2/(2c^2) - \\
&\boldsymbol{a}t^3\dot{\boldsymbol{a}}\cdot\boldsymbol{r}/(12c^2) + \boldsymbol{a}t^4\dot{\boldsymbol{a}}\cdot\boldsymbol{v}/(12c^2) - t\boldsymbol{v} - 5/16\,\dot{\boldsymbol{a}}t^2v^2\boldsymbol{r}\cdot\boldsymbol{v}/c^4 + \\
&7/48\,\dot{\boldsymbol{a}}t^3v^2\boldsymbol{a}\cdot\boldsymbol{r}/c^4 + 5/24\,\dot{\boldsymbol{a}}t^3\boldsymbol{a}\cdot\boldsymbol{v}\boldsymbol{r}\cdot\boldsymbol{v}/c^4 - \\
&\dot{\boldsymbol{a}}t^4\boldsymbol{a}\cdot\boldsymbol{a}\boldsymbol{r}\cdot\boldsymbol{v}/(32c^4) - \dot{\boldsymbol{a}}t^4\boldsymbol{a}\cdot\boldsymbol{r}\boldsymbol{a}\cdot\boldsymbol{v}/(8c^4) - t\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{v}\boldsymbol{r}\cdot\boldsymbol{v}/(4c^4) - \\
&3/8\,tv^2\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{r}/c^4 - t^2\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{a}\boldsymbol{r}\cdot\boldsymbol{v}/(8c^4) + t^2\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{r}\boldsymbol{a}\cdot\boldsymbol{v}/(2c^4) + \\
&t^2\boldsymbol{v}\dot{\boldsymbol{a}}\cdot\boldsymbol{v}\boldsymbol{r}\cdot\boldsymbol{v}/(8c^4) + 3/16\,t^2v^2\boldsymbol{v}\dot{\boldsymbol{a}}\cdot\boldsymbol{r}/c^4 + t^3\boldsymbol{v}\boldsymbol{a}\cdot\dot{\boldsymbol{a}}\boldsymbol{r}\cdot\boldsymbol{v}/(8c^4) - \\
&5/24\,t^3\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{r}\dot{\boldsymbol{a}}\cdot\boldsymbol{v}/c^4 - 7/24\,t^3\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{v}\dot{\boldsymbol{a}}\cdot\boldsymbol{r}/c^4 + \\
&7/96\,t^4\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{a}\dot{\boldsymbol{a}}\cdot\boldsymbol{r}/c^4 + t^4\boldsymbol{v}\boldsymbol{a}\cdot\dot{\boldsymbol{a}}\boldsymbol{a}\cdot\boldsymbol{r}/(48c^4) + 3/8\,v^2\boldsymbol{v}\boldsymbol{r}\cdot\boldsymbol{v}/c^4 - \\
&\dot{\boldsymbol{a}}t^2\boldsymbol{r}\cdot\boldsymbol{v}/(4c^2) + \dot{\boldsymbol{a}}t^3v^2/(4c^2) + \dot{\boldsymbol{a}}t^3\boldsymbol{a}\cdot\boldsymbol{r}/(12c^2) - \\
&\dot{\boldsymbol{a}}t^4\boldsymbol{a}\cdot\boldsymbol{v}/(12c^2) - t\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{r}/(2c^2) - tv^2\boldsymbol{v}/(2c^2) + \boldsymbol{v}\boldsymbol{r}\cdot\boldsymbol{v}/(2c^2) + \\
&t^2\boldsymbol{v}\boldsymbol{a}\cdot\boldsymbol{v}/(2c^2) + t^2\boldsymbol{v}\dot{\boldsymbol{a}}\cdot\boldsymbol{r}/(4c^2) - t^3\boldsymbol{v}\dot{\boldsymbol{a}}\cdot\boldsymbol{v}/(4c^2)
\end{aligned}$$

$$t' = t + 3/8\,tv^4/c^4 - v^2\boldsymbol{r}\cdot\boldsymbol{v}/(2c^4) + tv^2/(2c^2) - \boldsymbol{r}\cdot\boldsymbol{v}/c^2.$$

The solution is only to order v^4, but not many powers of velocity are meaningful in periodic solutions without also carrying the $\ddot{\boldsymbol{a}}$ terms. The invariant quantity for this solution is $\boldsymbol{r}\cdot\boldsymbol{r} - c^2t^2$. The $\boldsymbol{a}$ and $\dot{\boldsymbol{a}}$ terms drop out. The solution is available in a form that can be imported into other computer programs in the SOM.

The v^5 solution contains terms that are not invariant, and using more integration steps does not help. If the trajectory is represented by a truncated Taylor series, the particle velocity will eventually exceed c, which is probably why there is no solution accurate to order v^5 for a jerked clock.

V. THE YANKED CLOCK

The solution to order $v^4 a^3 \dot{a}^2 \ddot{a}^1$ is shown in the SOM. The solution is invariant. It is equivalent to a Lorentz transform followed by three consecutive space rotations.

The solution was obtained with a ninth order extrapolation equation, with $n = 1$. Using a larger value for n makes no difference at all in the final solution, showing that the calculations are of first order.

Including the constant term, there are only five degrees of freedom in the five vectors representing the trajectory in our frame of reference. Our place in space and time has no special significance for other observers. The equations require additional degrees of freedom if they are to lead to a global solution that is valid for distant observers other than ourselves.

The extrapolation equation is similar to a Taylor series in $\frac{1}{n}$. Including the constant term, the 9^{th} order formula has 10 degrees of freedom. The solution has the form of a polynomial in t. The largest power of t is 8, so it has only 9 degrees of freedom. Despite its complexity, the extrapolation equation has only one more degree of freedom than the solution.

Due to excessive software execution time, it has not been determined if an invariant solution accurate to order v^5 exists.

VI. DISCUSSION

These calculations are equivalent to integrating from standstill in the frame of reference of the clock (or particle). The calculations indicate that solutions based on an infinity of infinitesimal Lorentz transforms are needlessly inefficient. A simpler equation with fewer degrees of freedom would suffice.

It is likely that there is a connection to the equations of the Lorentz group[3,5]. Those equations may provide a better theoretical basis for the calculations, along with the possibility of more compact solutions. The equations behave like a four dimensional version of the Taylor theorem.

The large powers of t in the solutions are not very important. Dropping them would have the effect of hiding the limitation to low velocities, which could simplify comparisons to analytical results. Analytical results do not normally have an explicit velocity limitation of less than c, but that does not necessarily mean that their range of validity is unbounded. When working in series form, there are some freedoms in choosing the powers that the variables are to be carried to.

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